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Precision determination of α_s and m_b from QCD sum rules for $b\overline{b}$.

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Abstract

The QCD sum rules for moments of production cross section of $b\overline{b}$ states in $e^+\,e^-$ annihilation are extremely sensitive to the values of m_b and α_s for moments of large order n. This enables one to extract from the existing data on Υ resonances the values of these parameters with a high precision by using a non-relativistic expansion in $1/n$. It is found that the sum rules fit the data with $\alpha_s^{\overline{MS}}(1\,GeV) = 0.336 \pm 0.011$ and $m_b = 4827 \pm 7 MeV$, where the estimate of the errors includes the theoretical uncertainty due to subleading in $1/n$ terms and the experimental uncertainty of the $e^+\,e^-$ annihilation cross section above the $B\overline{B}$ threshold. The found value of α_s, when evolved in two loops up to the Z mass, gives $\alpha_s^{\overline{MS}}(M_Z) = 0.109 \pm 0.001$. The b quark mass m_b corresponds to the 'on shell' value appropriate for one-loop perturbative calculations.

1 Introduction

Precision tests of the Standard Model, which are becoming possible due to experimental technique, call for a better precision in understanding the parameters of the Model as we see it now. In particular the value of the b quark mass enters as a parameter in the predictions for the decays of B hadrons, and its understanding is necessary e.g. for a precise determination of the weak mixing element V_{cb} from the data on inclusive decays of B hadrons. The value of α_s determined from a global fit to the data of the LEP experiments at the Z peak: $\alpha_s^{\overline{MS}}(M_Z) = 0.125 \pm 0.005 \pm 0.002$ (see e.g. in [1] and also in [2]) is argued[3, 4] to be meaningfully higher than what one finds from extrapolation to the Z mass scale of the values of the QCD coupling found in a number of analyses at low energies. If confirmed with further improvement of experimental accuracy and of the theoretical understanding, this mismatch in the values of α_s may signal a contribution of a new physics to decays of Z [3, 4]. It should be noted however, that at present there exists a variety of low-energy estimates of α_s, with various degree of compatibility with the LEP value, which reflects the present uncertainty in the extraction of α_s from the low-energy data. (The most recent and extensive review of various fits of the value of α_s is given by Hinchliffe[5, 6].)

The purpose of this paper is to present one more determination of α_s from the low-energy phenomenology, namely from an analysis of the QCD sum rules for the cross section of production of the $b\bar{b}$ hadronic states $X_{b\bar{b}}$ in $e^+ e^-$ annihilation. Simultaneously the same analysis yields a precision determination of the mass parameter m_b. For the states $X_{b\bar{b}}$ the QCD sum rules[7] relate the integral moments of the physically measured quantity $R_b = \sigma(e^+ e^- \to X_{b\bar{b}})/\sigma(e^+ e^- \to \mu^+ \mu^-)$ of the form $\int ds\, R_b(s)/s^{n+1}$ to theoretically calculable derivatives of the vacuum polarization by the vector current $(\bar{b}\,\gamma_\mu\, b)$. For high enough n the moments are saturated by the lowest vector resonances: Υ's, and are essentially not sensitive to the uncertainty of the cross section in the continuum above the $B\overline{B}$ threshold.

It has been noticed long ago[7, 8] that for large n the theoretical calculation of the moments within the perturbation theory contains as a parameter $\alpha_s \sqrt{n}$ rather than α_s, which in a dispersive calculation corresponds to dominance of the near-threshold quark-antiquark dynamics at typical velocity $v \approx 1/\sqrt{n}$. So that $\alpha_s \sqrt{n} \approx \alpha_s/v$ is the familiar Coulomb parameter. Therefore at large n the Coulomb effects should be explicitly summed up. On the other hand, this behavior implies that high moments are very sensitive to the value of α_s and thus this value can be extracted with high accuracy, even though the experimental input is not very precise. Also, by dimension, the n-th moment depends on m_b^{2n}, which explains the high sensitivity at large n to the quark mass.

The limiting factor in considering moments with high n is the growth of the relative contribution of the non-perturbative terms. The first one, appearing in the sum rules for heavy quarks, is proportional to the so-called gluon vacuum condensate $\langle \alpha_s G^2_{\mu\nu} \rangle$ [9]. This term grows approximately as n^3 relative to the perturbative one[9, 8]. However in the case of the sum rules for the $b\overline{b}$ states its magnitude is still within about 1% at $n = 20$ (and becomes rapidly more important at higher n). In what follows the range of moments from $n = 8$ to $n = 20$ will be considered, where the non-perturbative contribution can be safely neglected and systematically the leading in $1/n$ approximation will be used, which, in particular, allows summation of the Coulomb terms $(\alpha_s \sqrt{n})^k$.

An analysis of the sum rules for the $b\overline{b}$ states along these lines was done quite some time ago[8] using then available data and a higher range of n: from $n \approx 25$ to $n \approx 40$, with taking into account the first non-perturbative term. Those estimates resulted in evaluating the mass parameter $m_b = 4.80 \pm 0.03 GeV$ and in an estimate of α_s: $\alpha_s = 0.30 \pm 0.03$ at a momentum scale of order 1 GeV. Here the analysis is refined by calculating the effects of running of the coupling α_s thereby precisely specifying the scale for α_s as a function of n. Also an uncertainty due to the next term in the $1/n$ expansion is estimated, which allows to consider a lower range of n, where the results are not affected by an uncertainty in the value of the gluon condensate.

One can present several reasons for this method of determining α_s being one of the most, if not the most, reliable. Firstly, it is fully justified within the short-distance QCD and does not rely on assumptions about local duality, as one has to, when considering rates at fixed energy, like, say, in the annihilation of heavy quarkonia into gluons, or even in more traditional processes like the total cross section of $e^+ e^-$ annihilation into hadrons at fixed energy, or the total hadronic decay rate of the τ lepton[1]. Secondly, since the parameter $\alpha_s \sqrt{n}$ in the considered range of n is of order one, the effects of the coupling are not just small corrections but in fact are dominating. Therefore the value of the coupling can be determined with a high precision. Thirdly, the moments with sufficiently high n are only very weakly sensitive to the poorly known cross section above the $B\overline{B}$ threshold. Thus even quite conservative assumptions about this cross section are sufficient for performing the analysis with an acceptable accuracy.

[1]The later method of determining α_s (see e.g. in [10]) was recently criticized by Shifman[4] on the basis of possible corrections to local duality in the timelike domain.

2 Sum rules

The QCD sum rules[7] under discussion arise from considering the vacuum polarization operator $P_b(q^2)$ induced by the electromagnetic current $j_\mu = Q_b\,(\overline{b}\,\gamma_\mu\,b)$ of the b quarks:

$$P_b(q^2) = \frac{-i}{3\,q^2}\int\, e^{iqx}\,\langle 0|T\,(j_\mu(x)\,j_\mu(0))\,|0\rangle \tag{1}$$

with $Q_b = -1/3$ being the electric charge of b quark. The imaginary part of $P_b(s)$ is related to the physically measurable quantity $R_b(s)$: $\mathrm{Im}P_b(s) = R_b(s)/12\pi$. Therefore using dispersion relation one can write the n-th derivative of $P_b(q^2)$ at $q^2 = 0$ in terms of the n-th integral moment of $R_b(s)$:

$$12\pi^2\,\left(\frac{d}{dq^2}\right)^n\,P_b(q^2)\Big|_{q^2=0} = \int\,R_b(s)\,\frac{ds}{s^{n+1}}\quad , \tag{2}$$

where the integral runs over all physical values of s where the cross section for production of states containing the $b\overline{b}$ quark pair in $e^+\,e^-$ annihilation is non-zero, i.e. it includes the Υ resonances and the continuum above the $B\overline{B}$ threshold. On the other hand the same derivatives as in eq.(2) can be calculated by methods of the short-distance QCD. In particular in perturbation theory this can be done by the usual calculation of Feynman graphs for $P_b(q^2)$, which in any finite order in α_s is equivalent to writing those derivatives in terms of the formal integrals with the value of $R_b(s)$ given by the perturbation theory in that order in α_s: $R_b^{pt}(s)$. In this way one arrives at the equations

$$\int\,R_b(s)\,\frac{ds}{s^{n+1}} = \int\,R_b^{pt}(s)\,\frac{ds}{s^{n+1}} + non{-}pert.\ terms \tag{3}$$

which express the QCD sum rules in this particular case. The left hand side of the sum rules contains a physically measurable quantity, while the right hand side is calculable theoretically including the first non-perturbative terms[7, 9].

2.1 Non-relativistic limit in the sum rules

One can notice that at large n the weight function $1/s^{n+1}$ rapidly decreases with energy, so that the integral is dominated by the lowest states of $b\overline{b}$. For the perturbation theory integral in eq.(3) this implies that at large n the integral is dominated by the states which are close to the quark-antiquark threshold $s_0 = 4m_b^2$. Defining E as the energy counted from the quark threshold, i.e. $s = (2m_b + E)^2$, one can write

$$\int\,R_b(s)\,\frac{ds}{s^{n+1}} = \int\,R_b\,\frac{2\,(2m_b+E)\,dE}{(2m_b+E)^{2n+2}} = \frac{1}{(4m_b^2)^n}\int\,R_b\,\exp\left(-\frac{E}{m_b}n\right)\,\frac{dE}{2m_b}\,\left(1+O\left(\frac{1}{n}\right)\right)\ , \tag{4}$$

thus effectively replacing the power weight function by an exponential. Moreover, it is clear that the integral is determined by the range of energy where $|E|/m_b \sim 1/n$, i.e. by the non-relativistic region near the threshold. If one also recalls the relation of the energy to the velocity v of each of the quarks (in the c.m. system): $E = m_b\, v^2$, one concludes that the relevant range of velocity is given by $v \sim n^{-1/2}$, so that the $1/n$ expansion of the integrals in the sum rules is equivalent to a non-relativistic expansion in v^2 [11].

Furthermore, in the non-relativistic region the spectral density of the electromagnetic current can be expressed through the spectral density of the non-relativistic operator $\delta(\mathbf{r})$, which is convenient to represent in terms of the matrix element $\langle 0|(H-E)^{-1}|0\rangle$ of the Green's function $\langle \mathbf{x}|(H-E)^{-1}|\mathbf{y}\rangle$ with the non-relativistic Hamiltonian H [11]:

$$R_b((2m_b+E)^2) = \left(1 - \frac{16\,\alpha_s}{3\pi}\right)\, \frac{18\pi\, Q_b^2}{m_b^2}\, \mathrm{Im}\langle 0|(H-E)^{-1}|0\rangle\ . \tag{5}$$

Here is also included the radiative correction, which comes from distances of order $1/m_b$ and represents a finite renormalization of the electromagnetic current at the threshold. This correction will be further discussed and quantified at the end of this section.

Substituting the expression (5) into the exponential integral in eq.(4) one finds a relation[8] of the integral moments to the matrix element $K(\tau) = \langle 0|\exp(-H\,\tau)|0\rangle$ of the Euclidean time propagation operator $\exp(-H\,\tau)$ at $\tau = n/m_b$:

$$\int R_b(s)\,\frac{ds}{s^{n+1}} = \left(1 - \frac{16\,\alpha_s}{3\pi}\right)\, \frac{18\pi^2\, Q_b^2}{(4m_b^2)^n\, m_b^3}\, K\left(\frac{n}{m_b}\right)\, \left(1 + O\left(\frac{1}{n}\right)\right)\ . \tag{6}$$

The latter representation in terms of $K(\tau)$ enables a detailed analysis of the effects of quark interaction in the non-relativistic domain both at the perturbative and the non-perturbative level.

2.2 Summing Coulomb effects

In the lowest order of perturbation theory, i.e. for free quarks the propagation function has the familiar form (see e.g. in [12], and it should also be taken into account that the reduced mass in the $b\overline{b}$ system is $m_b/2$):

$$\langle \mathbf{x}|\exp(-H_0\,\tau)|\mathbf{y}\rangle = \left(\frac{m_b}{4\pi\,\tau}\right)^{3/2}\, \exp\left(-\frac{m_b}{4\,\tau}(\mathbf{x}-\mathbf{y})^2\right)\ , \tag{7}$$

so that

$$K_0(\tau) = \left(\frac{m_b}{4\pi\,\tau}\right)^{3/2}\ , \tag{8}$$

which reproduces the large n limit of the moments of $R_b(s)$ calculated from the simple loop of Fig.1.

The QCD interaction of quarks via gluons in the leading in $v^2 \sim 1/n$ approximation reduces to a Coulomb-like potential

$$V(r) = -\frac{4\alpha_s}{3r} \; . \tag{9}$$

The perturbation theory expansion in this potential generates an expansion for $K(\tau)$ in powers of the parameter $\alpha_s \sqrt{m_b \tau} = \alpha_s \sqrt{n}$. Since we are aiming at the region of n where this parameter is of order one, the interaction (9) should be taken into account exactly rather than perturbatively. In practice this amounts to finding the propagation function $K(\tau)$ in the Coulomb problem. In terms of Feynman graphs this is equivalent to summing the diagrams of Fig.2 in the non-relativistic region. The time-dependent propagator in the Coulomb field can be found[8, 13] by the inverse Laplace transform of the energy-dependent Green's function at negative energy $E = -k^2/m_b$, which can be derived from a solution of the Schrödinger equation in terms of the singular at $z = 0$ confluent hypergeometric function $U(a,\, b;\, z)$:

$$\langle r| \left(H + \frac{k^2}{m_b} \right)^{-1} |0\rangle = \frac{m\, k}{2\pi}\, e^{-kr}\, \Gamma(1-\lambda)\, U(1-\lambda,\, 2,\, 2kr) = \frac{m\, k}{2\pi}\, e^{-kr} \int_0^\infty e^{-2krt} \left(\frac{1+t}{t} \right)^\lambda dt \; , \tag{10}$$

where $\lambda = 2m_b\, \alpha_s/(3k)$ is the Coulomb parameter for the potential (9). Setting $r = 0$ in the integral representation in eq.(10) gives a divergent integral. However the divergent term does not depend on k and thus does not affect the inverse Laplace transform in the variable k^2/m. The result for $K(\tau)$ has the form[8] $K(n/m_b) = K_0(n/m_b)\, F(\gamma)$, where $\gamma = 2\alpha_s \sqrt{n}/3$ is the Coulomb parameter in the time-dependent representation, and

$$F(\gamma) = 1 + 2\sqrt{\pi}\gamma + \frac{2\pi^2}{3}\gamma^2 + 4\sqrt{\pi} \sum_{p=1}^{\infty} \left(\frac{\gamma}{p} \right)^3 \exp\left[\left(\frac{\gamma}{p} \right)^2 \right] \left[1 + \mathrm{erf}\left(\frac{\gamma}{p} \right) \right] \; , \tag{11}$$

where, as usual, $\mathrm{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-t^2)\, dt$. One can readily see that the series in eq.(11) is well converging, so that this expression can be used for practical calculation. The function $F(\gamma)$ sums all the terms of the form $(\alpha_s \sqrt{n})^k$ in the time dependent propagator and thus in the sum rules. One can also easily recognize that in the formal limit $n \to \infty$ the function $F(\gamma)$ is determined by the first term in the sum in eq.(11): $F(\gamma) \to 8\sqrt{\pi}\gamma^3 \exp(\gamma^2)$ which is exactly the contribution of the lowest Coulomb bound state[2]. In the analysis of the sum rules for the $b\overline{b}$ system we will be restricted by the region, where γ is of order 1. In

[2]It should be noticed however that at finite γ each p-th term in the sum in eq.(11) receives contribution both from the p-th Coulomb bound state and from the continuum above the threshold.

connection with this it can be noticed that numerically the function F is large in this region: $F(1) = 49.1243\ldots$, which explains the high sensitivity of the sum rules to the value of α_s.

2.3 Effects of running α_s

In the previous discussion the effects of running of the renormalized coupling α_s were ignored. These have to be included in order to be able to relate the moments of R_b to the value of α_s at a specified scale. Naively, one would estimate that at a velocity v the typical momentum flowing through the Coulomb gluons in the graphs of Fig.2 is of order $m_b v$, which in the τ domain translates into the expectation that the normalization for α_s scales as $\sqrt{m_b/\tau} = m_b/\sqrt{n}$. However the Coulomb effects substantially modify this estimate in the region where α_s/v, or equivalently γ, is not small. This can be seen from the simple fact that in the formal limit $\gamma \to \infty$ the function $K(\tau)$ is determined by the lowest Coulomb level, which has the intrinsic scale given by the Bohr momentum $k_B = 2\alpha_s m_b/3$. Therefore one should expect that the proper normalization scale for α_s in the propagation function $K(\tau)$ is given by $\sqrt{m_b/\tau}\, h(\gamma)$, i.e. $K(\tau) = K_0(\tau)\, F(\gamma(\sqrt{m_b/\tau}\, h(\gamma)))$ where the function $h(\gamma)$ describes the Coulomb effects on the scale and the overall normalization of $h(\gamma)$ depends on the renormalization scheme for α_s. In particular the dominance of the lowest Coulomb level at large γ implies that in this limit $h(\gamma) \propto \gamma$.

To quantify the effects of running α_s we adopt the $\overline{MS}$ scheme, in which the α_s in the interaction potential in eq.(9) is[14] $\alpha_s^{\overline{MS}}(\kappa/r)$ with $\kappa = \exp(-C - 5/6)$ and where $C = 0.5772\ldots$ is the Euler constant. When expressed in terms of $\alpha_s^{\overline{MS}}(\mu)$ normalized at a fixed scale μ, the effect of running α_s in the potential (9) is reduced to a modification of the potential of the form:

$$V(r) = -\frac{4\alpha_s^{\overline{MS}}(\mu)}{3r} - \frac{4\alpha_s}{3r}\frac{b\,\alpha_s}{2\pi}\ln\frac{\mu r}{\kappa} + \ldots\ , \tag{12}$$

where $b = 9$ is the first coefficient in the QCD β function. (We deal here with the typical scales about 1 GeV, so that the appropriate number of active quark flavors in the β function is $n_f = 3$.) On the other hand if the function $K(\tau)$ is written in terms of $\gamma(\mu) = 2\alpha_s^{\overline{MS}}(\mu)\sqrt{n}/3$, the same first order effect in the running results in the following modification for the function $F(\gamma)$:

$$F = F(\gamma(\mu)) + \gamma\,\frac{dF}{d\gamma}\,\frac{b\,\alpha_s}{2\pi}\ln\left(\frac{\mu\sqrt{n}}{h(\gamma)\,m_b}\right)\ . \tag{13}$$

Finally, the modification of the Coulomb function $F(\gamma)$ is clearly the result of the modification of the potential in eq.(12). To relate the two expressions it is technically convenient to first

find the modification of the energy-dependent Green's function by the perturbation of the potential as in eq.(12) and then to find the corresponding change of the time-dependent propagator by the inverse Laplace transform. For the energy-dependent Green's function the correction has the form:

$$\delta\langle 0|\left(H+\frac{k^2}{m_b}\right)^{-1}|0\rangle = \frac{4\alpha_s}{3}\,\frac{b\,\alpha_s}{2\pi}\,\frac{m^2\,k^2}{\pi}\int dt\,du\,dr\,r\,e^{-2kr(t+u+1)}\,\left(\frac{(1+t)(1+u)}{t\,u}\right)^{\lambda}\,\ln\frac{\mu r}{\kappa}\;, \tag{14}$$

where each of the integrals runs from zero to infinity. The integration over r can be easily done explicitly, which however is not as simple for the integration over t and u. However the integral can be expanded in powers of λ and transformed into the τ representation term by term, which generates an expansion of the modification $\delta F(\gamma)$ of the function F in powers of γ. (In doing this transformation one only needs the transformation rule for a generic term $1/k^p$: $1/k^p \to \tau^{p/2-1}/(m_b^{p/2}\,\Gamma(p/2))$.) This procedure works for all terms of the expansion of the integral in eq.(14) in powers of λ, except for the first one with λ^0, which contains a manifestly divergent integral. However this term can be easily calculated directly in terms of $K(\tau)$ using the free evolution function in eq.(7). This amounts to calculating the integral

$$\frac{m_b^3}{16\,\pi^2}\int_0^{\tau}\frac{d\tau_1}{\tau_1^{3/2}\,(\tau-\tau_1)^{3/2}}\int dr\,\exp\left(-\frac{m_b\,r^2\,\tau}{4\,\tau_1\,(\tau-\tau_1)}\right)\ln r = \frac{m_b^2}{8\pi\,\tau}\,\left[\ln\left(\frac{1}{2}\sqrt{\frac{\tau}{m_b}}\right)-\frac{C}{2}\right]\;. \tag{15}$$

Collecting all terms and equating the modification of the function $F(\gamma)$ in eq.(13) to that calculated from the perturbation of the potential in eq.(12) we finally find the function $h(\gamma)$ in the following form of expansion in powers of γ:

$$\ln h(\gamma) = \ln(2\kappa) + \frac{2\sqrt{\pi}}{F'(\gamma)}\left(\frac{C}{2}+2\sum_{p=1}^{\infty}a_p\,\gamma^p\right)\;, \tag{16}$$

where $F' = dF/d\gamma$ and the coefficients a_p are given by the integrals

$$a_p = \int_0^1\int_0^1 dy\,dz\frac{\{\psi(p/2)/2-\psi(2)-\ln\left[(1-y)(1-z)/(1-y\,z)\right]\}\,(-\ln yz)^p}{p!\,\Gamma(p/2)\,(1-y\,z)^2}\;, \tag{17}$$

where $\psi(x) = d\ln\Gamma(x)/dx$ and the integration variables y and z are related to those in eq.(14) by the obvious substitution: $y = t/(1+t)$, $z = u/(1+u)$. The integrals (17) for the coefficients a_p can be successfully calculated numerically. The convergence of the expansion in eq.(16) is however somewhat slow. In particular, for calculating $h(g)$ with four digit accuracy at $\gamma = 1$ one has to keep 21 first terms in the expansion. (In the numerical analysis, described in the next section, were used 30 terms of the expansion.)

Thus we find that the normalization point for α_s in the parameter γ in eq.(11) is itself determined by the value of γ. Therefore in order to take into account the running of the coupling constant in one loop one has to determine the value of γ in eq.(11) by solving the equation

$$\gamma = \gamma \left(\frac{m_b}{\sqrt{n}}\, h(\gamma) \right) \ , \tag{18}$$

which can be easily done numerically by iterations.

2.4 Short-distance radiative correction

Thus discussed radiative effects arise from distances, which are parametrically larger than m_b^{-1} by a factor $\sqrt{n}\, h(\gamma)$, which are the characteristic distances in the non-relativistic Coulomb dynamics of quarks at a velocity $v \sim 1/\sqrt{n}$. There is however a radiative effect, which comes from the distances of order m_b^{-1} and is associated with correction to the electromagnetic vertex. (A discussion of this point in QED can be found in the textbook [15].) To quantify this effect let us consider the graphs shown in Fig.3 for the calculation in up to the one-loop order of the spatial part of the vector current $(\overline{b}\,\gamma_i\, b)$ exactly at the $b\overline{b}$ threshold, i.e. for the quark and the antiquark being at rest in the c.m. system. (It is only the spatial part of the current which is non vanishing in the c.m. kinematics.) The sum of the graphs can be written as

$$(\overline{b}\,\gamma_i\, b)\, \left(1 + \frac{\alpha_s}{3\,\pi} J \right) \ , \tag{19}$$

where the one-loop correction term J after integrating over the angular variables can be written in the form of an integral over the Euclidean k^2 of the virtual gluon:

$$J = \int_0^\infty \left(\frac{4}{t^{3/2}} + w(t) \right) dt \ , \tag{20}$$

where $t = k^2/m_b^2$ and the integrand contains the infrared-singular term $4/t^{3/2}$ and the regular part $w(t)$, for which the explicit expression is

$$w(t) = \frac{4}{3\, t^{3/2}\, \sqrt{t+4}} \left(6 - 2\, t + 2\, t^2 + t^3 - 3\, \sqrt{t+4} - t^{5/2}\, \sqrt{t+4} \right) \ . \tag{21}$$

The infrared divergence of the singular term corresponds to the singularity of the first Coulomb correction $\propto \alpha_s/v$ at velocity $v = 0$, and this term is in fact the one accounted for in the Coulomb problem calculation[15]. Therefore the additional radiative correction is described only by the regular part with $w(t)$. The integral with the $w(t)$ from eq.(21) can be readily done:

$$\int_0^\infty w(t)\, dt = -8 \ , \tag{22}$$

which results in the well known expression for the radiative correction to the matrix element squared, used in eq.(6). What is important here is that the integral (22) is determined by virtual momenta of order m_b therefore in contrast with the Coulomb terms this radiative correction depends on α_s normalized at a scale of order m_b. A precise specification of this scale in the $\overline{MS}$ scheme requires a two loop calculation of the form factor at the threshold, which to the best of my knowledge is so far absent. In view of this we have to resort to evaluating the scale within the Brodsky - Lepage - Mackenzie (BLM) scheme[14], which technically amounts to averaging $\ln t$ with the weight function $w(t)$ determining the one-loop correction. For this average we find

$$\langle \frac{1}{2}\,\ln t\rangle \equiv \frac{1}{2}\,\frac{\int_0^\infty\, w(t)\,\ln t\,dt}{\int_0^\infty\, w(t)\,dt} = \frac{3}{8}\,. \tag{23}$$

By the BLM prescription the normalization point for α_s is then given by $e^{3/8}m_b$ in the V scheme (fixed by the Coulomb potential in the momentum space) and by $e^{-5/6}\,e^{3/8}m_b = e^{-11/24}m_b \approx 0.632\,m_b$ in the $\overline{MS}$ scheme. Therefore we conclude that within the BLM method the appropriate value of α_s in the short-distance radiative correction in eq.(6) is $\alpha_s^{\overline{MS}}(0.632\,m_b)$.

3 Numerical analysis

Summarizing the discussion of the previous section, the large n perturbative formula for the moments of $R_b(s)$, including the effects of running coupling and of the $O(\alpha_s)$ radiative correction, can be written as

$$\int\, R_b(s)\,\frac{ds}{s^{n+1}} = \left(1 - \frac{16\,\alpha_s^{\overline{MS}}(0.632m_b)}{3\pi}\right)\,\frac{\sqrt{\pi}\,9\,Q_b^2}{(4m_b^2)^n\,4\,n^{3/2}}\,F\left(\gamma(m_b\,h(\gamma)/\sqrt{n})\right)\,\left(1+O\left(\frac{1}{n}\right)\right)\,, \tag{24}$$

where the function $F(\gamma)$ is given by eq.(11) and its argument is found by solving the equation (18) with the function h determined by eqs.(16) and (17). Further perturbative corrections to the expression (24) are at least as small as $O(\alpha_s^2)$ with no additional enhancement in n. On the other hand the region of the parameters to be considered is where $\alpha_s\sqrt{n} = O(1)$. Thus the uncertainty due to further radiative corrections is parametrically the same as that from the $1/n$ terms.

Another class of corrections to eq.(24) goes beyond perturbation theory and is associated with non-perturbative properties of the QCD vacuum. These corrections grow with n, and the first one, due to the gluon condensate, is well known. In the leading $1/n$ approximation

this correction amounts[8] to replacing $F(\gamma)$ in eq.(24) by the expression

$$F(\gamma)\left(1-\xi(\gamma)\,\frac{n^3}{m_b^4}\,\langle\frac{\pi\,\alpha_s}{72}\,G^2_{\mu\nu}\rangle\right)\ , \tag{25}$$

where the function $\xi(\gamma)$ has a rather complicated form[8], which however can be approximated in the region $\gamma \leq 1.5$ by simple exponent: $\xi(\gamma) \approx \exp(-0.8\gamma)$. Using this approximation and the value[9] of the gluon condensate $\langle\frac{\alpha_s}{\pi}G^2\rangle \approx 0.012\,GeV^4$, one can estimate that the relative magnitude of the non-perturbative term does not exceed about 1% for $n \leq 20$. For this reason the present analysis is restricted to moments with $n \leq 20$ and the non- perturbative term is completely ignored.

On the lower end the range of n is restricted by the applicability of the $1/n$ expansion. Another practical restriction is that the value of R_b is very poorly known experimentally above the $B\overline{B}$ threshold. To keep this uncertainty well suppressed in the moments it is desirable to choose larger values of n.

In the present analysis as the experimental inputs are used the masses and the $e^+\,e^-$ widths with their experimental errors for the four lowest Υ resonances, as given by the Particle Data Tables[16]. In connection with this it can be reminded that the value of $R(s)$ for a narrow resonance can be written as

$$R_{Res}(s) = \frac{9\pi}{\overline{\alpha}^2}\,\Gamma(Res \to e^+\,e^-)\,M_{Res}\,\delta(s-M^2_{Res})\ , \tag{26}$$

where $\overline{\alpha}$ contains the renormalization of the electromagnetic α at energy near the mass of the resonance:

$$\overline{\alpha}^2 = \frac{\alpha^2}{|1-P_{em}(M^2_{Res})|^2} \tag{27}$$

with $P_{em}(q^2)$ being the full electromagnetic vacuum polarization operator. In the numerical analysis here we use the estimate $\overline{\alpha}^2 = 1.07\alpha^2$ at energy in the region of Υ resonances. This correction turns out to be quite essential in the numerical fits.

As to the cross section above the continuum threshold, i.e. above the $\Upsilon(4S)$, a conservative assumption is adopted here that the value of $R_b(s)$ is equal to $(1 \pm 0.5)/3$ starting from the mass of $\Upsilon(4S)$. I believe that the assigned error of 50% well covers the possible uncertainties in the contribution of the continuum cross section to the considered moments.

For a fit by the formula in eq.(24) these input values were used to calculate the "experimental" left hand side of eq.(24) for n = 8, 12, 16 and 20. The choice of these particular values is determined, in addition to the discussed restrictions on the range of n, by the requirement that the chosen moments are sufficiently representative of the statistically independent input data. In other words, if the values of n were chosen in a shorter range with smaller

spacing they would essentially represent less statistically independent values than there are experimental inputs. Quantitatively this choice is determined by the condition number of the covariance matrix of the values of the moments with respect to the dispersion of inputs.

The actual fit was done in several different ways as regards handling of the uncertainty due to the $O(1/n)$ terms in eq.(24) and the uncertainty in the continuum cross section. The effect of the former terms was parametrized as a factor $(1 + c/n)$ in the right hand side of eq.(24) and the value of the coefficient c was either treated as a fit parameter, or fixed at $c = 0$. Then the difference in the fit values of α_s and m_b in these two calculations gives the estimate of the uncertainty due to the unknown $O(1/n)$ terms. The 50% uncertainty in the cross section in the continuum was treated either as an additional statistical error, or the extreme cases were taken as fixed. For the α_s all fits were done in terms of $\alpha_s^{\overline{MS}}(1\,GeV)$.

The results of the fit, where the uncertainty in the continuum cross section is treated as a statistical error are the following. A three parameter fit for α_s, m_b and c yields: $\alpha_s^{\overline{MS}}(1\,GeV) = 0.336 \pm 0.006$, $m_b = 4827 \pm 4\,MeV$ and $c = -0.59 \pm 0.19$ with $\chi^2 = 0.8$ at the minimum corresponding to the central values of the fit parameters. If c is fixed at $c = 0$ and a two parameter fit is performed, the minimum of $\chi 2$ is reached at $\alpha_s^{\overline{MS}}(1\,GeV) = 0.325$ and $m_b = 4280\,MeV$. However the value of χ^2 at the minimum is $\chi^2 = 7.2$, thus it would not be reasonable to ascribe statistical errors to the fit parameters in this case. However, the difference of the central values for α_s and m_b in the two fits gives the measure of the uncertainty due to the $O(1/n)$ term, which thus leads to the estimate of the errors:

$$\begin{aligned} &\alpha_s^{\overline{MS}}(1\,GeV) = 0.336 \pm 0.011 \\ &m_b = 4827 \pm 7\,MeV\ . \end{aligned} \tag{28}$$

A plot, illustrating these two fits, is shown in Fig. 4.

If the continuum cross section is fixed at the lower extreme bound: $R_b(s) = 0.5/3$ at $\sqrt{s} > M(4S)$, the result of a three parameter fit is $\alpha_s^{\overline{MS}}(1\,GeV) = 0.338 \pm 0.005$, $m_b = 4828 \pm 4\,MeV$, $c = -0.71 \pm 0.16$ with $\chi^2 = 0.1$ at the minimum, which values are within the range given by eq.(28). However, if c is fixed at $c = 0$ the two parameter fit gives a statistically unacceptable minimal value $\chi^2 = 424$ at $\alpha_s^{\overline{MS}}(1\,GeV) = 0.313$ and $m_b = 4809\,MeV$. Setting the possible continuum cross section at the upper extreme value: $R_b(s) = 1.5/3$ does not give at all a statistically acceptable fit for either three or two parameters. (In the former case one finds $\chi^2 = 43$ at the minimum at $\alpha_s^{\overline{MS}}(1\,GeV) = 0.307$ and $m_b = 4808MeV$, while a two parameter fit gives a totally unreasonable minimal value $\chi^2 = 800$.) As a result we conclude that with all the uncertainty in the continuum cross section and in the $O(1/n)$ terms in the sum rules (24) taken into account the existing data on Υ resonances are only compatible with

values of α_s and m_b, which lie in the range given by eq.(28).

4 Discussion

In the derivation of the sum rules in eq.(24) only a one-loop running of α_s is taken into account. Therefore it might be possibly argued that the result for α_s found in the present analysis is not appropriate for a precision two- or three-loop evolution to higher scales, in particular for comparison with other estimates at the Z mass scale. To answer to this argument it is instructive to evaluate the actual interval of the normalization scale over which the α_s evolves in eq.(24) when n changes in the considered range, i.e. from $n = 8$ to $n = 20$. Using the central fit values for α_s and m_b in eq.(28), we find that the normalization point $\mu(n) = m_b\, h(\gamma)/\sqrt{n}$ changes from $\mu(8) = 1.028\, GeV$ down to $\mu(20) = 0.99\, GeV$ (for reference: the corresponding values of the Coulomb parameter γ are $\gamma = 0.625$ for $n = 8$ and $\gamma = 1.007$ at $n = 20$). Therefore change in $\ln \mu$ is only about 0.04, so that higher loop effects in the β function would be well less than the error for α_s in eq.(28). Naturally, a comparison of the present result with other estimates of α_s at different scales involves evolution of the coupling at large intervals of the normalization scale and should be done using the higher loop effects. Considering the evolution in two loops with the starting value of $\alpha_s^{\overline{MS}}(1\, GeV)$ in eq.(24), one finds:

$$\begin{aligned}
&\Lambda_{\overline{MS}}^{(3)} = 258 \pm 14\, MeV\ , \\
&\alpha_s^{\overline{MS}}(m_\tau) = 0.254 \pm 0.006\ , \\
&\alpha_s^{\overline{MS}}(m_b) = 0.185 \pm 0.003\ , \\
&\alpha_s^{\overline{MS}}(M_Z) = 0.109 \pm 0.001\ ,
\end{aligned} \tag{29}$$

where $\Lambda_{\overline{MS}}$ is defined as the position of the singularity in formal numerical solution of the two-loop evolution equation for α_s. These derived values of α_s are significantly lower than the central values of those found from the τ decay[10]: $\alpha_s^{\overline{MS}}(m_\tau) = 0.33 \pm 0.03$, and from the LEP data[1]: $\alpha_s^{\overline{MS}}(M_Z) = 0.125 \pm 0.005 \pm 0.002$, although there is a compatibility at the level of about 3σ due to larger errors of the latter determinations. On the other hand the value of α_s is in a very good agreement with the one determined from the three-gluon decay rates of the Υ resonances, which corresponds to $\alpha_s^{\overline{MS}}(M_Z) = 0.108 \pm 0.001$ (statistical error only) and which is often doubted as being subject to unknown non-perturbative and relativistic corrections, as is discussed in [5]. I believe that the approach used in this paper is intrinsically more accurate, than determining the value of α_s from small corrections. The

reliability of the considered method can be further improved by experimentally measuring the cross section of $e^+ e^-$ annihilation above the $\Upsilon(4S)$ up to about 12 GeV c.m. energy and by a theoretical analysis of the $1/n$ and α_s^2 terms.

The value of the quark mass in eq.(28) requires some specification, in view of the intrinsic uncertainty of the quark mass in non-perturbative QCD[17]. Apart from summation of the Coulomb terms, the sum rules in eq.(24) contain the QCD radiative effects of the first order in α_s. The on shell mass of a heavy quark is defined in any finite order of perturbation theory and enter as a formal parameter in a calculation to that order. In this sense the mass parameter m_b is the on shell mass appropriate for calculations at the one-loop radiative level in QCD. Naturally, the derived on shell mass in perturbation theory depends on the order in α_s in which the calculation is done. This is because a perturbative calculation is justified inasmuch as it is determined by the short-distance dynamics and thus it is in fact sensitive to an off-shell value of the quark mass $m(\mu)$ at short distances, which is a combination of the on-shell mass, the coupling constant $\alpha_s(\mu)$ and the momentum scale μ corresponding to the distance scale involved in the problem. In particular, for a heavy quark and $\mu \ll m$ the first-order relation is $m(\mu) = m - a\,\alpha_s(\mu)\,\mu$, where the constant a is scheme dependent as the off-shell mass $m(\mu)$ is, unlike the on-shell mass m. Therefore the on-shell mass derived from a short-distance calculation should correlate with the value of the coupling $\alpha_s(\mu)$. This correlation is conspicuously present in the considered here analysis of the sum rules (24), and can be clearly seen on the plot of Fig. 4. Within the described numerical analysis, one can find that an uncorrelated with α_s mass parameter is $m_b^* = m_b - 0.56\,\alpha_s^{\overline{MS}}(\mu)\,\mu$ for $\mu = 1\,GeV$, which is determined with a very high statistical accuracy: $m_b^* = 4639 \pm 2\,MeV$, which is due to the fact that the sum rules in the considered range of n are sensitive to dynamics at distances approximately 1 GeV^{-1}. In view of the scheme dependence of the off-shell mass the precise implications of this numerical observation are not quite clear.[3]

5 Acknowledgements

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[3]It is however interesting to note, in this connection, that the numerical value 0.56 of the coefficient a is very close to the value $a = 16/(9\pi) \approx 0.567$, which enters the definition of the off-shell mass by the amplitude of Thomson scattering[18].

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Figure Captions

Figure 1. The lowest order graph for vacuum polarization by the vector current of b quarks.

Figure 2. The type of graphs corresponding to summation of all powers of the Coulomb parameter α_s/v in the vacuum polarisation in the non-relativistic domain near the threshold. Dashed lines denote Coulomb gluons.

Figure 3. The graphs, describing up to the order α_s the matrix element for production of quark and antiquark by the vector current exactly at the threshold.

Figure 4. The contour plot illustrating the fit of the sum rules to the data. (The uncertainty in the continuum cross section above the threshold is treated as a statistical error.) The ellipse is the one standard deviation contour for $\alpha_s^{\overline{MS}}$ and m_b (in MeV) for the three parameter fit at the optimal value of c: $c = -0.59$. The heavy dot corresponds to the minimum of χ^2 in the two parameter fit.

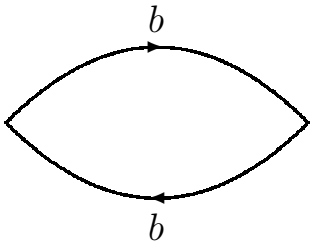

Fig. 1

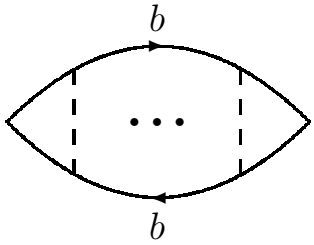

Fig. 2

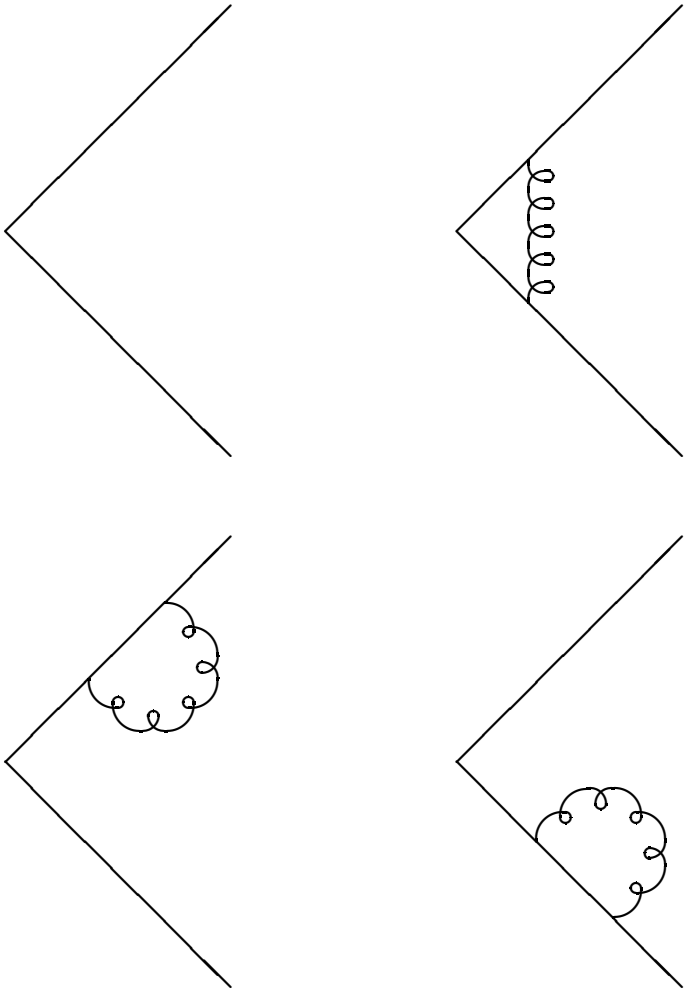

Fig. 3

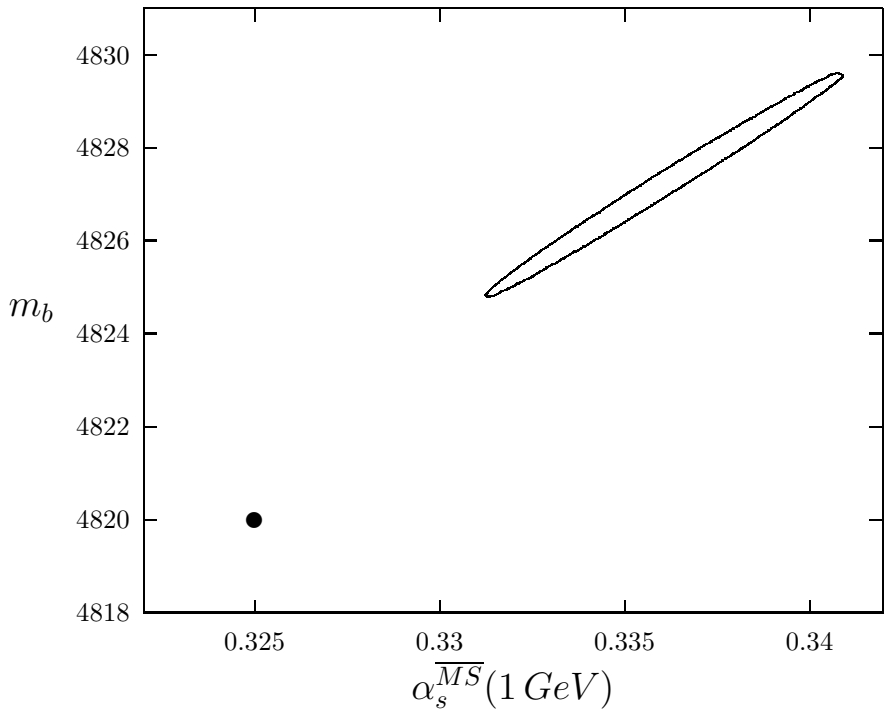
4830
4828
4826
4824
4822
4820
4818
m_b
0.325
0.33
0.335
0.34
$\alpha_s^{\overline{MS}}(1\,GeV)$

Fig. 4